Graph Theory [US06CMTH05]

(Syllabus effective from June, 2012)

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Unit : 4

CONTENTS

First and second isomorphisms , Planar graphs , Kuratowski's Two graphs , Different representations of a planar graphs , Detection of Planarity geometric and combinatorial dual.

4.1. Definition. First isomorphism

Let the process of splitting (hence duplicating) a cut-vertex into to two vertices to produce two disjoint subgraphs be known as Ôperation-1.

If two graphs G_1 and G_2 become isomorphic to each by repeatedly applying the Ôperation-1 on the graphs then they are called **1-isomorphic** to each other.
The disconnected graph in the figure 1 is 1-isomorphic to the graph in the figure **??** as it can be obtained by applying operation-1 at the cut-vertices a and b.

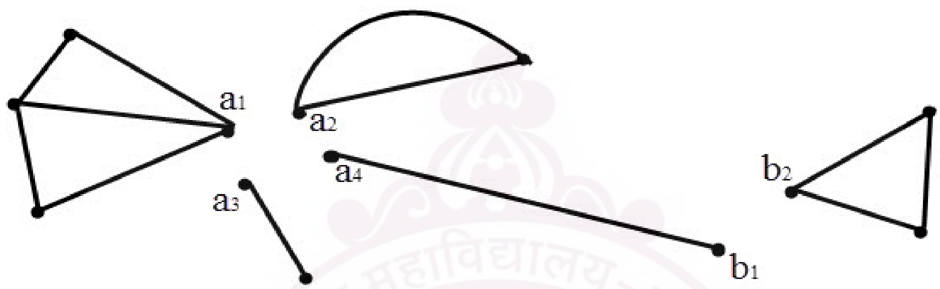


Figure 1. Applying Operation-1 on the graph in figure **??**

4.2. THEOREM. If G_1 and G_2 are two 1-isomorphic graphs then prove that rank of G_1 = rank of G_2 and the nullity of G_1 = nullity of G_2.

Proof : Let G_1 and G_2 be two 1-isomorphic graphs.
Suppose in G_1 there are n vertices and e edges. We know that as result of performing the operation of spliting a cut-vertex into two duplicate vertices one more vertex is added to the graph along with one more component of the graph.

If by performing such operations m times on G_1 we can obtain G_2 then there will be m more vertices and m more components in G_2 than G_1.
Therefore, number of vertice in G_2 is $n+m$ and number of components in G_2 is $k+m$

Hence,

$$\begin{aligned}\text{Rank of } G_2 &= \text{Number of vertices in } G_2 - \text{Number of components in } G_2\\ &= (n+m)-(k+m)\\ &= n-k\end{aligned}$$

$\therefore$ Rank of G_2 = Rank of G_1

Also, above operations neither add nor remove any edge of G_1. So the number of edges in 1-isomorphic graphs G_1 and G_2 are same.

Therefore,

$$\begin{aligned}\text{Nullity of } G_2 &= \text{Number of edges in} G_2 - \text{Rank of } G_2 \\ &= \text{Number of edges in } G_1 - \text{Rank of } G_1 \\ &= \text{Nullity of } G_1 \\ \therefore \text{Nullity of } G_2 &= \text{Nullity of } G_1\end{aligned}$$

4.3. Example. Second isomorphism

First we define following operations

Operation-1 :

In a seperable graph split (hence duplicate) a cut-vertex into two vertices to produce two disjoint subgraphs.

Operation-2 :

In a 2-connected graph G (Figure: 2(a)) let x and y be a pair of vertices whose removal from the graph will leave the resultant graph disconnected.
Split x and y both into two pairs of vertices x_1, x_2 and y_1, y_2 respectively and let the resultant subgraphs be g and $\overline{g}$, such that x_1 and y_1 remain with g and x_2 and y_2 remain with $\overline{g}$ (Figure: 2(b)).
By merging x_1 with y_2 and x_2 with y_1 rejoin the two subgraphs g and $\overline{g}$ (Figure: 2(d))

Two graphs G_1 and G_2 are said to be 2-isomorphic with each other if they become isomorphic after going under the operation-1 or operation-2 or both the operations any number of times.

The graphs in Figure: 2(a) and Figure: 2(d) are 2-isomorphic.

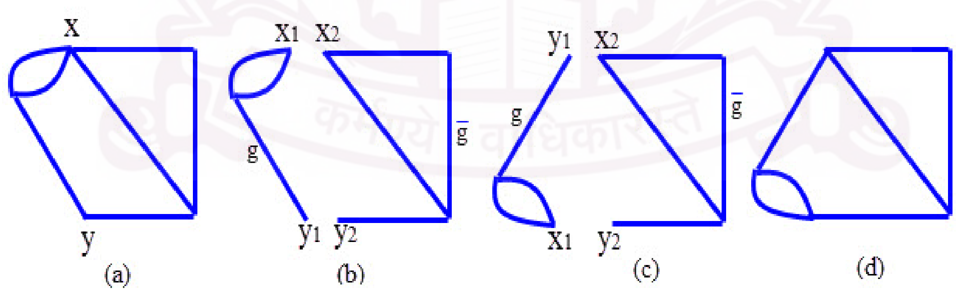


Figure 2. 2-isomorphic graphs [(a) and (d)]

4.4. Definition. Circuit correspondence

Circuit correspondence :

Two graphs G_1 and G_2 are said to have Circuit correspondance if they meet the following conditions

(i) There is one-to-one correspondance between the edges of G_1

and G_2 and
(ii) one-to-one correspondance between the circuits of G_1 and G_2 such that a circuit in G_1 formed by certain edges of G_1 has a corresponding circuit in G_2 formed by corresponding edges in G_2, and vice versa.

4.5. THEOREM. Prove that 2-isomorphic graphs have circuit correspondence.

Proof : Let G_1 and G_2 be 2-isomorphic with each other.
Therefore, they become isomorphic after going under the operation-1 (Spliting a cut-vertex) or operation-2 (Spliting two vertices that disconnect a 2-connected graph and cross-merging them) or both the operations any number of times, if required.

Now, Operation-1 splits a vertex into two leaving the edges intact. Therefore the circuits are retained in their original form after a graph undergoes operation-1.

Also if x and y are two vertices in a 2-connected graph whose removal from the graph disconnects the graph into two subgraphs g and $\overline{g}$,
then after performing Operation-2 at these points there are following three possilibities for any circuit τ in the graph.

1. τ consists of edges in g
2. τ consists of edges in $\overline{g}$
3. τ consists of edges in g and $\overline{g}$ both.

In first two cases the circiuit ramains unchanged.

In the last case τ has all the original edges except the edge connecting x and y has flipped after operation-2. But the flipping of the edge does not change the edge correspondance. Therefore after undergoing operation-2 also a graph retains a circuit in its original form.

Thus, there is a one-to-one correspondance between the circuits of G_1 and G_2.

4.6. Definition. Planar graph

A graph is said to be **Planar** if there exists a geometric representation of G which can be drawn on a plane such that no two of its edges intersect.

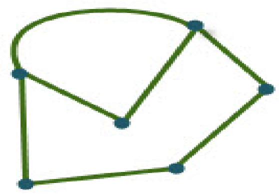


FIGURE 3. Geometric representation of a planar grpah

4.7. DEFINITION. Embedding

The drawing of a geometric representation of graph G on any surface that no edges intersect is called **Embedding** of the graph G on the surface.

4.8. DEFINITION. Nonplanar graph

A graph is said to be **Nonplanar** if it cannot be drawn on a plane without a crossover between atleast one pair of of its edges.

4.9. EXAMPLE. Using geometric arguments prove that a complete graph of 5 vertices is non planar.

Proof : **Kuratowski's first graph:**
A complete graph with 5 vertices is called Kuratowski's first graph. The graph is generally denoted by K_5.

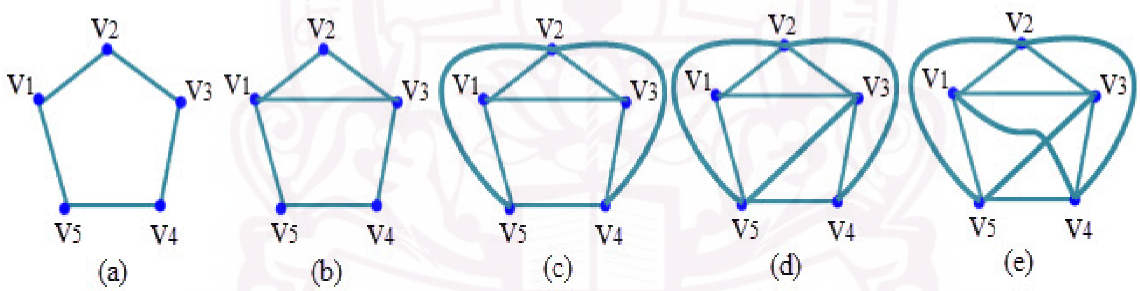


FIGURE 4. Constructing Kuratowki's first graph (K_5)

Now we shall show that the graph is nonplanar using impossibility of its embedding on a plane surface.

Let v_1, v_2, v_3, v_4 and v_5 be the vertices of the graph. As it is a complete graph with 5 vertices each vertex must be connected directly through an edge with each of the remaining 4 vertices. We shall try to make such connections avoiding any crossover of the edges.

Clearly there must be a circuit through all the edges. We draw this circuit as a pentagon with the vertices of the graph (Figure: 4(a)). The pentagon divides the plane of the drawing paper into two regions, one inside and the other outside. Each vertex is connected with two neighbouring vertices and needs to be connected with rest of the vertices.

Let us draw an edge inside the pentagon to connect the vertices v_1 and v_3 (Figure: 4(b)). This edge divides the pentagon interior into two parts such that v_4 and v_5 lie on the opposite side of the part containg v_2.

Now we draw two edges lying in the exterior of the pentagon which connect v_2 with v_4 and v_5 and then we draw an edge in the interior to connect v_3 with v_5 (Figure: 4(c) and (Figure: 4(d))). So far there no edge that crosses over any other edge.

Finally we need to connect v_1 with v_4 directly with an edge. As they lie in two different closed regions formed by some of the edges that we have drawn. , it is impossible to draw an edge to conncet the two vertices without crossing over atleast one of the existing edges (Figure: 4(e)). Similarly any other attempt of embedding the graph in a plane fails.

Thus, the graph is nonplanar as it cannot be embedded in a plane.

4.10. Example. Using geometric arguments prove that $K_{3,3}$is non-planar.

Proof : Kuratowski's second graph:
A regular connected graph with 6 vertices and 9 edges is called Kuratowski's second graph. The graph is generally denoted by $K_{3,3}$.

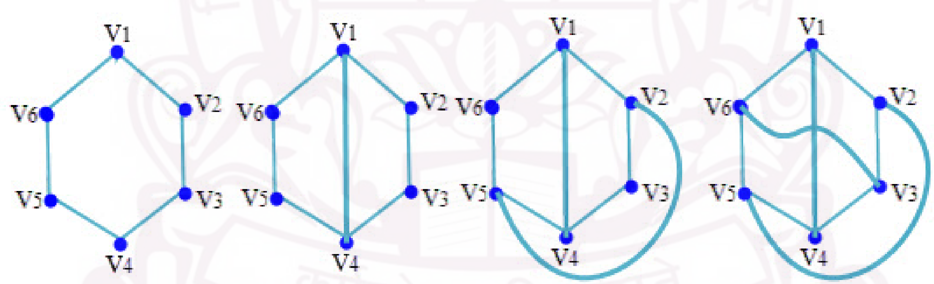


Figure 5. Constructing Kuratowki's second graph ($K_{3,3}$)

Now we shall show that the graph is nonplanar using impossibility of its embedding on a plane surface.

Let v_1, v_2, v_3, v_4, v_5 and v_6 be the vertices of the graph. The degree of each vertex is 3. Therefore, to embed the graph in a plane each vertex is to be connected directly, through an edge, with three of the remaining vertices. We shall try to make such connections avoiding any crossover of the edges.

Clearly there must be a circuit through all the edges. We draw this circuit as a hexagon with the vertices of the graph (Figure: 5(a)). The hexagon divides the plane of the drawing paper into two regions, one inside and the other outside. Each vertex is connected with two neighbouring vertices and needs to be connected with one of the other vertices.

Let us draw an edge inside the pentagon to connect the vertices v_1 and v_4 (Figure: 5(b)). This edge divides the pentagon interior into two parts such that v_2 and v_3 lie on the opposite side of the part containg v_5 and v_6.

Now we draw an edge lying in the exterior of the pentagon which connect v_2 with v_5 (Figure: 5(c)). So far there no edge that crosses over any other edge.

Finally we need to connect v_3 with v_6 directly with an edge. As v_6 lies outside the closed region containing v_3 , it is impossible to draw an edge to conncet the two vertices without crossing over atleast one of the existing edges (Figure: 5(d)). Similarly any other attempt of embedding the graph in a plane fails.

Thus, the graph is nonplanar as it cannot be embedded in a plane.

4.11. Example. Discuss Kurtowski's two graphs

Solution :

(A) Kuratowki's first graph (K_5) (B) Kuratowki's second graph ($K_{3,3}$)

Kuratowski's first graph:
A complete graph with 5 vertices is called Kuratowski's first graph. The graph is generally denoted by K_5.

Kuratowski's second graph:
A regular connected graph with 6 vertices and 9 edges is called Kuratowski's second graph. The graph is generally denoted by $K_{3,3}$.

The two graphs have the following properties in common.
1. Both the graphs are regular
2. Both the graphs are nonplanar
3. Removal of one edge or one vertex makes each graph planar.
4. Both the graphs are simplest nonplanar graphs as K_5 is a nonplanar graph with smallest number of vertices and $K_{3,3}$ is a nonplanar graph with smallest number of edges.

[**A**] Discuss Kurtowski's First graphs

Solution.

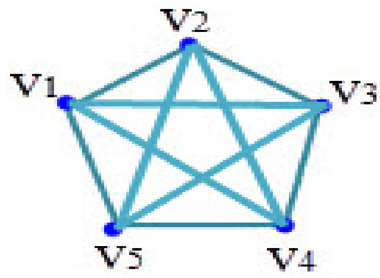


FIGURE 7. Kuratowki's first graph (K_5)

Kuratowski's first graph:
A complete graph with 5 vertices is called Kuratowski's first graph. The graph is generally denoted by K_5.

A regular connected graph with 6 vertices and 9 edges is called Kuratowski's second graph. The graph is generally denoted by $K_{3,3}$. The graph has following properties.

1. It is a regular nonplanar graph
2. Removal of one edge or one vertex makes the graph planar.
3. It is one of the simplest nonplanar graphs as it is a nonplanar graph with smallest number of vertices.

[**B**] Kurtowski's Second graphs

Solution.

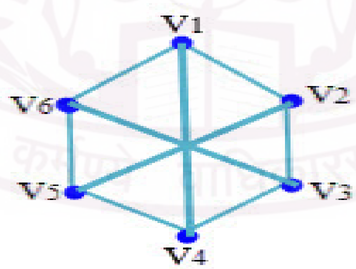


FIGURE 8. Kuratowki's second graph ($K_{3,3}$)

Kuratowski's Second graph ($K_{3,3}$):
A regular connected graph with 6 vertices and 9 edges is called Kuratowski's second graph. The graph is generally denoted by $K_{3,3}$. The graph has following properties.

1. It is a regular nonplanar graph
2. Removal of one edge or one vertex makes the graph planar.
3. It is one of the simplest nonplanar graphs as it is a nonplanar graph with smallest number of edges.

4.12. Definition. Region (face) and Infinite region

A geometric representation of a planar graph divides the plane into some closed areas bounded by edges and some unbounded open areas. Each closed area is called a **Region** or a **Face** of the graph and each unbounded area is called an **Infinite Region** of the graph.

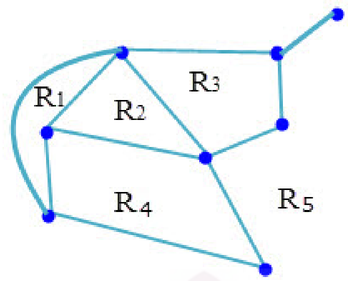


Figure 9. Regions (Faces) of a graph

In the graph shown in the figure 9 has four regions R_1, R_2, R_3, R_4 and an infinite region R_5.

4.13. THEOREM. State and prove Euler's theorem for planar graphs.

Proof : We know that adding a self-loop or a parallel edge adds one region to the graph and at the same time the value of e is increased by one. Also removal of an edge that does not form a boundary of any region either increases or decreases e and n both by 1 simultenously. Therfore such removal leaves $e - n$ unchanged. Thus, it is sufficient to prove the result for a simple planar graph.

Now, a planar graph can be drawn such that each region is a polygon. Therefore we can a draw a simple planar graph as a polygonal net.

Suppose the polygonal net representing a simple planar graph G consists of f regions (faces) and let k_p be the number of p-sided regions in the graph. Since each edge is on a boundary of exactly two regions, we have,

$$3k_3 + 4k_4 + 5k_5 + \cdots + rk_r = 2e$$

where k_r is the number of polygons with maximum number of edges. Also,

$$k_3 + k_4 + k_5 + \cdots + k_r = f$$

Now, the sum of all angles subtended at each vertex in the polygonal net is $2\pi n$

Moreover, the sum of all interior angles of a p-sided polygon is $\pi(p-2)$ and the sum of the exterior angles is $\pi(p+2)$. Therefore we can also calculate the sum of all the angles subtended at each vertex in the polygonal net as the grand sum of all the interior angles of

$f - 1$ regions plus the sum of all the exterior angles of the infinite region.
Therefore,

$$\begin{aligned} 2n\pi &= \pi(3-2)k_3 + \pi(4-2)k_4 + \cdots + \pi(r-2)k_r + 4\pi \\ &= \pi(3k_3 + 4k_4 + 5k_5 + \cdots + rk_r) - 2(k_3 + k_4 + k_5 + \cdots + k_r) + 4\pi \\ &= \pi(2e - 2f) + 4\pi \\ &= 2\pi(e - f + 2) \end{aligned}$$

$\therefore n = e - f + 2$

Therefore, the number of regions in G is given by $f = e - n + 2$.

4.14. THEOREM. For a simple connected planar graph with n-vertices , e-edges ($e > 2$) and f-regions prove the following.

(i) $e \geqslant \frac{3}{2}f$ (ii) $e \leqslant 3n - 6$ (iii) $e \leqslant 2n - 4$ where $e \geqslant 4$

Proof : Let G be a simple planar graph with n vertices and e edges, where $e > 2$.
(a) Since an edge of a graph is common to exactly 2 regions and a region has atleast 3 edges we have,

$$2e \geqslant 3f \quad \text{- - - (1)}$$

$$\therefore e > \frac{3}{2}f$$

(b) Also using the Euler's formula for a planar graph we have,

$$f = e - n + 2$$

Substituting in $2e \geqslant 3f$ we get,

$$2e \geqslant 3(e - n + 2)$$

$$2e \geqslant 3e - 3n + 6$$

$$3n - 6 \geqslant e$$

(c) Now, if we assume that each region in a graph is bounded by atleast 4 regions then we get

$$2e \geqslant 4f$$

Again using Euler's formula for a planar graph we have,

$$f = e - n + 2$$

Substituting in $2e \geqslant 4f$ we get,

$$2e \geqslant 4(e - n + 2)$$

$$e \geqslant 2(e - n + 2)$$

$$e \geqslant 2e - 2n + 4$$

$$2n - 4 \geqslant e$$

4.15. Example. Using Euler's theorem prove that Kuratowski's first and second graphs are non-planar.

Proof : In a graph every region is bounded by atleast three edges and each edge belongs to exactly two regions. Therefore, if there are e edges, n vertices and f faces then

$$2e \geqslant 3f$$

Using Euler's formula for a planar graph, $f = e - n + 2$ we get,

$$2e \geqslant 3(e - n + 2)$$

Therefore for a planar graph,

$$3n - 6 \geqslant e$$

Now, Kuratowski's first graph is a complete graph with $n = 5$ vertices and $e = \dfrac{5(5-1)}{2} = 10$ edges.

If the Kuratowski's first graph K_5 were planar then $n = 5$ and $e = 10$ must satisfy above inequality.
But taking $n = 5$ and $e = 10$ in the inequality $3n - 6 \geqslant e$ we get,

$$3(5) - 6 > 10$$

$$9 > 10$$

Which is not possible.
Therefore K_5 does not satisfy a necessary condition (the inequality) for coplanarity.

Hence, Kuratowski's first graph K_5 is not coplanar.

Also, Kuratowski's second graph is a regular connected graph with six vertices and nine edges.

Moreover, no region in this graph can be bounded with fewer than four edges.

Hence, if this graph were planar, we would have $2e > 4f$, and substituting for f from Euler's formula $f = e - n + 2$, we get,

$$2e > 4(e - n + 2)$$

If the Kuratowski's second graph $K_{3,3}$ were planar then $n = 6$ and $e = 9$ must satisfy this inequality. Taking $n = 6$ and $e = 9$ in the inequality we get,

$$2.9 > 4(9 - 6 + 2)$$

or

$$18 > 20$$

, Which is not possible. Hence Kuratowski's second graph is also nonplanar.

4.16. EXAMPLE. Describe the method of Detection of Planarity of a graph using elementary reduction.

Solution : To determine planarity of a graph we can use the following steps known as Elementary Reductions.
STEP:1 A seperable graph is planar if and only if each of its block is planar. Similarly a disconnected graph is planar if and only if each of its components is planar. Therefore, first we divide a graph into non-seperable blocks and obtain the set

$$G = \{G_1, G_2, \ldots, G_k\}$$

where $G_1, G_2, \ldots, G_k$ are nonseparable blocks of G.

STEP:2 Remove all self loops as they do no affect planarity.

STEP:3 Parallel edges also do not affect planarity. Therefore from each group of parallel edges we remove all edges except any one in each group.

STEP:4 Elimination of a vertex of degree two by merging two edges in a series does not affect planarity. Therefore eliminate all the edges in a series.

Repeated application of steps 3 and 4 reduce a graph to its most minimised form with same planarity.

If each of the blocks reduces to a planar graph then the graph G is planar otherwise it is nonplanar.

Using elementary reductions a block is reduced to a single edge in the figure 10

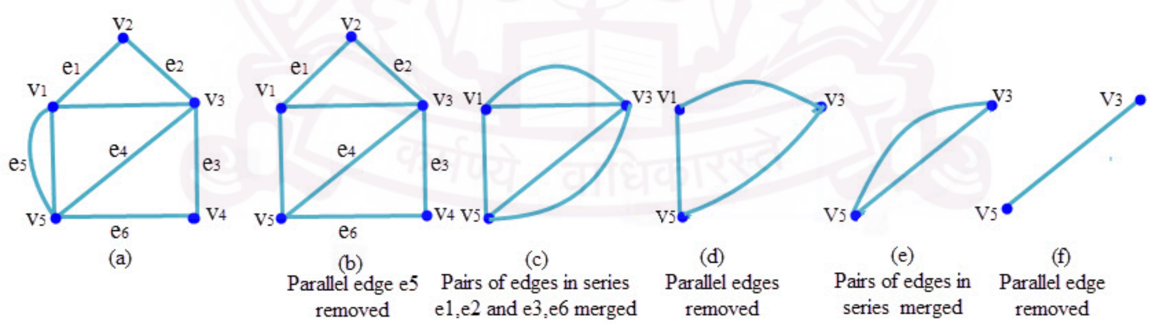


FIGURE 10. Detection of Planarity using Elementary Reductions

4.17. DEFINITION. Homeomorphic graphs

<u>Homeomorphic Graphs</u>
Two graphs are said to be **Homeomorphic** if one graph can be obtained from the other by creating edges in a series or by merging edges in series.

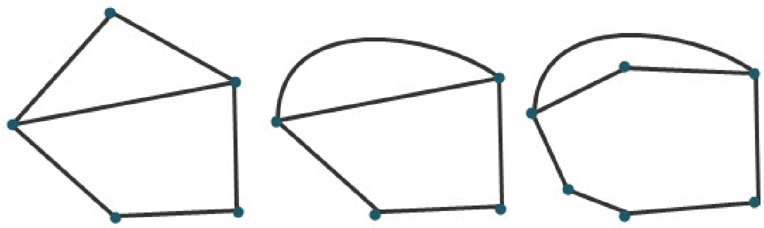


FIGURE 11. Homeomorphic Graphs

4.18. THEOREM. Prove that a necessary condition for a graph G to be a planar graph is that G does not contain either of a Kuratowski's two graphs or any graph homeomorphic to either of them.

Proof : Suppose a graph G is a planar graph.
If possible suppose G has a subgraph that is either one of the Kuratowski's graphs or it is homeomorphic to one of them.

In any case the subgraph is nonplanar and has no embedding in a plane. Consequently the graph G also cannot have an embedding in a plane
This is a contradiction as each planar graph has an embedding in a plane. Therefore, our suppposition is wrong.

Hence, a planar graph cannot contain either of a Kuratowski's two graphs or any graph homeomorphic to either of them.

4.19. DEFINITION. Geometric dual

Let G be graph whose plane reprsentation has k faces $F_1, F_2, \ldots, F_k$.

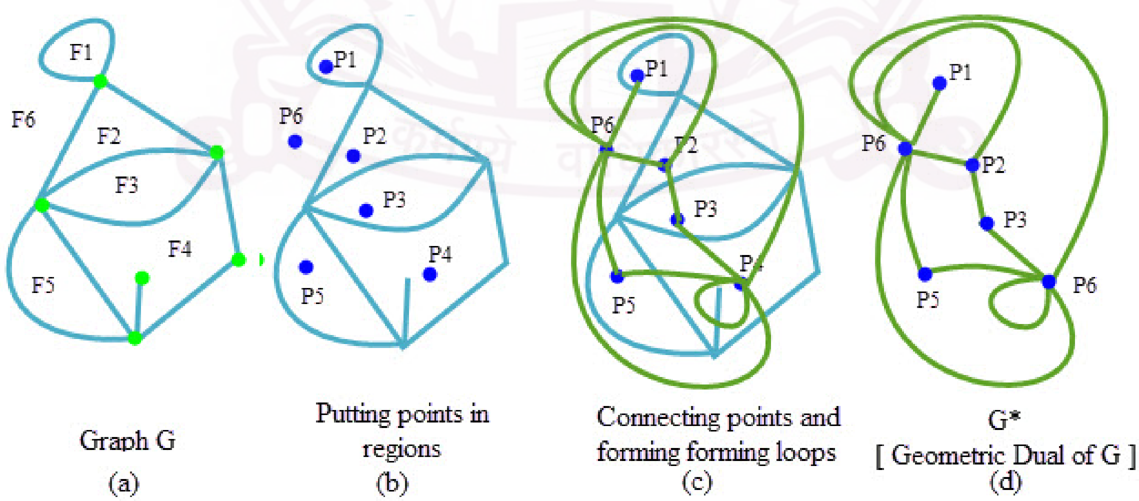


FIGURE 12. Homeomorphic Graphs

A graph obtained as follows is called the geometric dual of G and usually it is denoted by $G*$.

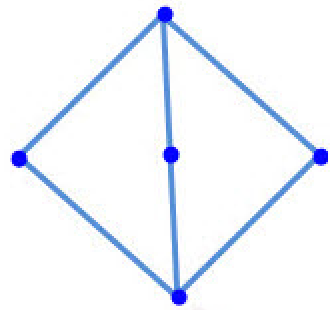
1. Place k points $P_1, P_2, \ldots, P_k$, one in each region.

2. If two regions F_i and F_j have a commin edge then draw a line segment joining the pair of points P_i and P_j that intersects the common edge. In case there are more than one common edges between the two regions then draw such line segments joining P_i and P_j for each of the common edges.

3. For an edge lying entirely in one region draw a selfloop at the point placed in that region intersecting the edge exactly once. We note that there is one to one corrspondance between the edges in G and its geometric dual $G*$. Also number of vertices in $G*$ is equal to the number of regions in G.

In the figure 12 the process of obtaining geometric dual of a graph G is shown.

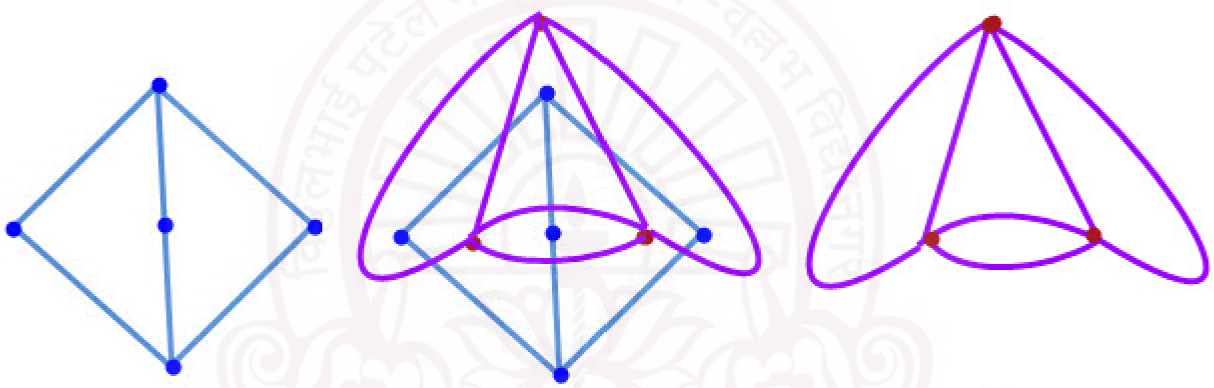
4.20. Example. Find geometric dual of the following graph



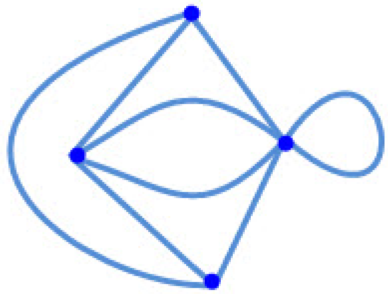
G_1

(1)

Solution :

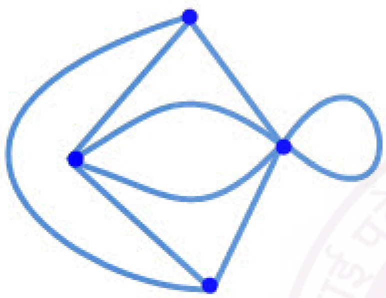


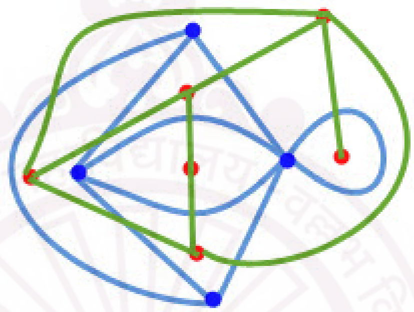
G_1 Finding the dual G_1^*

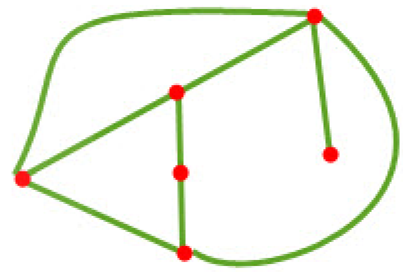


(2)

Solution :

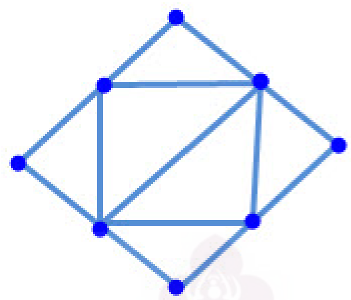

G


Finding the dual

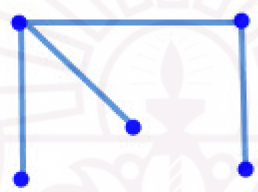

G*

Exercise : 0.1

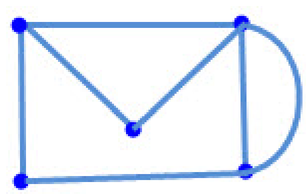
Find geometric dual of the following graph



(1)



(2)



(3)

4.21. Example. Give an example to show that dual of dual of a graph may not be isomorphic to the original graph.

Solution : Let G be a disconnected planar graph as shown in the figure 13

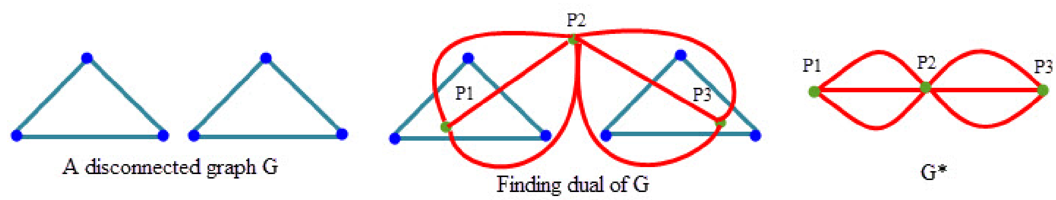


Figure 13. Isomorphic graphs

Last graph in the figure 13 is the dual of G.

Now, we find G^{**}, the dual of G^{*} as shown in the figure 14

Figure 14. Dual of the dual of G

Here the number of vertices in G is 6 and in G^{**} it is 5. As for isomorphism between two graphs it is necessary that they have same number of vertices, the graphs G and G^{**} are not isomorphic.

4.22. Example. Give an example to show that two isomorphic graphs may not have isomorphic duals.

Solution : Consider the graphs in the figure 15. The graphs are isomorphic with each other as there is a one-to-one correspondance between the vertices of the graphs and one-to-one correspondance between the edges such that end vertices of the edges are also in one-to-one correspondance.

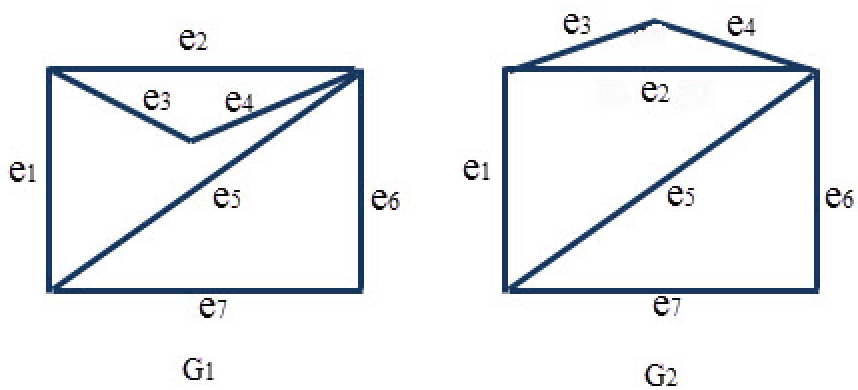


Figure 15. Isomorphic graphs

First, we find the geometric dual G_1^* of G_1 (figure 16)

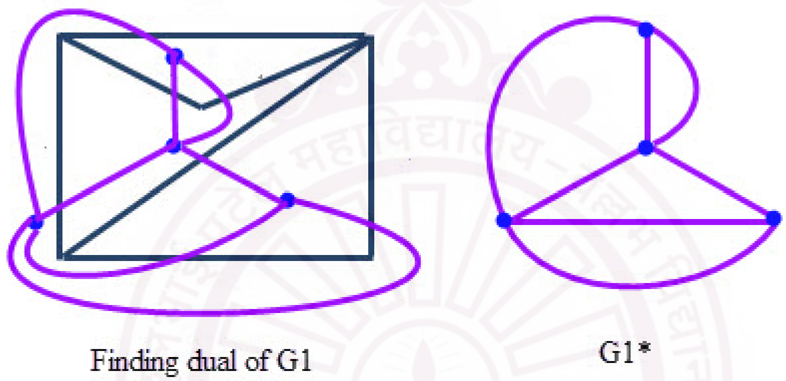


Figure 16. Dual of G_1

Next, we find the geometric dual G_2^* of G_2 (figure 17)

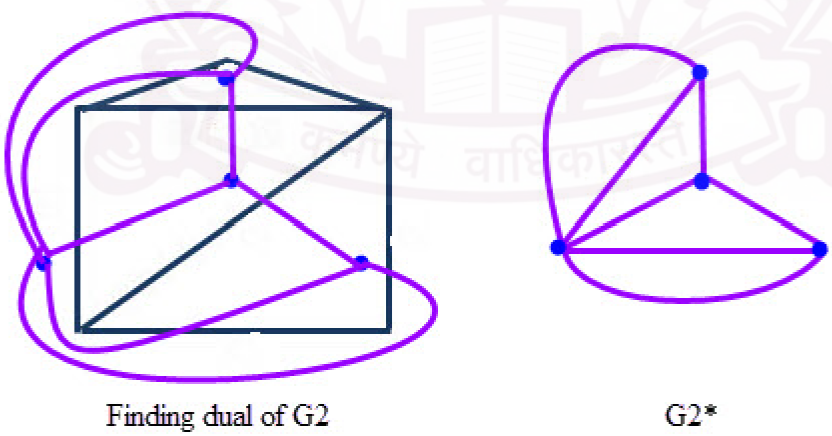


Figure 17. Dual of G_1

In G_1^* there are two vertices of degree 4 whereas there is no vertix of degree 4 in G_2^*.
As it is necessary for two isomorphic graphs to have same number of vertices with a given degree, the duals of the isomorphic graphs G_1 and G_2 are not isomorphic.

4.23. THEOREM. Prove that a neccessary and sufficient condition for two planar graphs G_1 and G_2 to be dual of each other is that there is one-one correspondence between the edges in G_1 and edges in G_2 such that a set of edges in G_1 forms a circuit *iff* the corresponding set in G_2 forms a cut-set.

Proof : First we show that the condition is necessary.
Let G^* be a plane representation of a planar graph G. Any circuit τ in G forms closed simple curve in a plane representation of G that divides the plane into two areas.
As each region in G corresponds a vertex in G^*, all the vertices of G^* are partitioned into two non-empty and mutually exclisive subsets, one inside τ and and other outside τ. Consequently the set of edges in τ^* corrsponding to τ is a cut-set in τ^*.
Similarly a set of edges S in G that corresponds to a cut-set S^* in G^* is a circuit in G.
Now we show sufficiency of the condition.

Suppose G is a planar graph and G' is a graph for which there is a one-to-one correspondance between the cut-sets of G and circuits of G' and vice versa.There is also one-to-onecorrespondance between the cutsets of G and circuits of its dual G^*.

Thus, there is exists a one-to-one correspondance between the circuits of G' and G^*. Therefore G' and G^* are 2-isomorphic.
Hence G' is a dual of G.

4.24. EXAMPLE. If G^* is dual of a graph G then describe a method of obtaining dual of a subgraph of G from G^*.

Solution : Let a be an edge in a planar graph G and a^* be corresponding edge in $G*$, the dual of G.

After deletion on a from G let us find the dual of $G - a$. If the edge a was a boundary of two regions in G then the two regions will merge into one in $G - a$.

Thus, the dual $(G - a)^*$ can be directly obtained from $G - a$ by directly deleting a^* and then merging its end vertices into one.

Now, if a is not on a boundary then a^* forms a self loop. In that case $G^* - a*$ is same as $(G - a)*$.

With successive application of this procedure we can obtain dual of a subgraph of G if its dual exists.

4.25. EXAMPLE. If G^* is dual of a graph G then describe a method of obtaining dual of a graph homeomorphic to G from G^*.

Method : Obtaining Dual of a graph Homeomorphic to a graph.

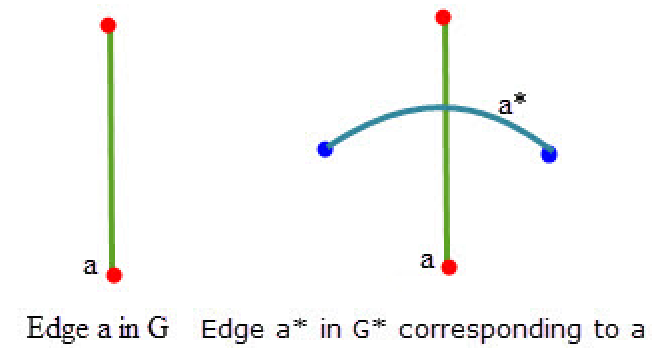


FIGURE 18

Let a be an edge in a planar graph G and a^* be corresponding edge in G^*, the dual of G.
Now addition of a vertex of degree two inbetween the edge a will create two edges in series out of a. As a result there will be an edge added to G^* parallel to a^*

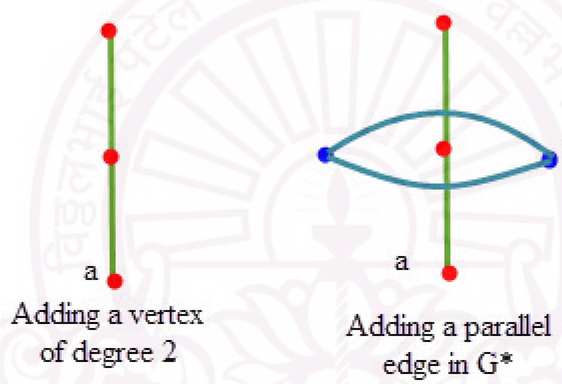


FIGURE 19

Similarly, if in G two edges in series are added then one of the corresponding parallel edges in G^* will be deleted.

Thus, if a graph G has a dual G^* then dual of any graph homeomorphic to G can be ontained directly from G^* by applying above process successively.

4.26. THEOREM. Prove that a graph has a dual *iff* it is planar.

Proof : As every planar graph has a dual, it remains to prove that if a graph has a dual then it is planar.
This we shall prove this by showing that a nonplanar graph cannot have a dual.

Let G be a nobplanar graph. Therefore, by Kuratowski's theorem, the graph contains K_5 or $K_{3,3}$ or has a subgraph homeomorphic to either of them.

Now for any graph to have a dual it is necessary that every subgraph of that graph has a dual. Therefore to show that G does not have a dual it is sufficient to show that neither of K_5 nor $K_{3,3}$ has a dual.

First we show that neither of $K_{3,3}$ does not have a dual.
If possible suppose $K_{3,3}$ has a dual, say D.
Now, every cut-set in a graph corresponds to a circuit in its dual and vice versa. As $K_{3,3}$ has no cut-set of two edges there is no circuit of two edges in D. This implies that D has no parallel edges. As every circuit in $K_{3,3}$ is of length 4 or 6, degree of every vertex in D is atleast 4. Since D has no parallel edges and degree of each of its vertex is atleast 4 , D must have atleast 5 vertices such that degree of each vertex is atleast 4. But then the number of edges in D must be $\dfrac{4 \times 5}{2} = 10$
This is a contradiction as $K_{3,3}$ has only 9 edges. Hence $K_{3,3}$ does not have a dual.

Finally we show that K_5 also does not have a dual.
If possible, suppose K_5 has a dual, say H.
We know that K_5 has
(1) 10 edges
(2) no pair of parallel edges
(3) no cut-set with 2 edges, and
(4) every cut-set has four or six edges.

Consequently, the dual H must have
(1) 10 edges
(2) no vertex with degree less than 3
(3) no pair of parallel edges and
(4) every circuit has length 4 and 6 only

Now, H contains a hexagon (a circuit with 6 edges) and not more than 3 edges can be added to a hexagon without creating a circuit of length 3 or a pair of parallel edges.
As H has neither a pair of parllel edges nor a circuit of length less than 4, the only possibility is that H has atleast 7 vertices. Moreover degree of each vertex must be 3. Therefore, H must have atleast 11 edges. This is a contradiction as H must have 10 edges only. Hence K_5 has no dual.
Thus, a nonplanar graph does not have a dual.

Hence, the theorem.